Markov Projections of the Voter Model

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A general framework of aggregation for the Voter Model (VM) is developed which exploits the symmetries of the agent network G. Depending on the symmetry group $Aut(G)$ of the graph, certain ensembles of agent configurations can be interchanged without affecting the probabilistic structure of the VM. These nodes can be aggregated into the same macro state and the dynamical process projected onto these states is still a Markov chain. The method informs about the complexity of a system introduced by heterogeneous interaction relations. In some cases the macro chain is solvable.

Agent–based models (ABMs) are from a formal point of view Markov chains [1, 2]. However, there is not too much to learn from this fact alone as the state space of the chains corresponding to those models (consisting of all possible configurations of agents) is tremendous. For the analysis of a model, therefore, a considerable reduction of the state space is required. In practice, the models are often re–formulated in terms of an aggregate variable (such as magnetization) so that all micro configurations with the same aggregate value are agglomerated. However, for inhomogeneous agent networks the dynamical process at a coarse–grained level obtained in this way is in general no longer a Markov chain [2]. In this case, the results are usually considered as an approximation and compared to simulations. Another possibility, however, is to refine the aggregation procedure such that the Markov property is not lost and try to solve the refined problem. In this Letter a systematic approach to aggregation is developed which exploits all the dynamical redundancies that have its source in the agent network on which the model is implemented.

We use the Voter Model (VM) to exemplify our approach (see Refs. [3–8]). The VM has its origin in the population genetics literature [3], but due to its interpretation as a model for opinion dynamics it has become a benchmark model in social dynamics. The reader is referred to Ref. [9] for a recent review of related models in population genetics and to Ref. [10] for an overview over the social dynamics literature.

Consider a set $\mathbf{N}$ of N agents, each one characterized by an individual attribute x_i which takes a value among two possible alternatives: $x_i \in \{\square, \blacksquare\}, i = 1, \ldots, N$. Let us refer to the set of all possible combinations of attributes of the agents as configuration space and denote it as $\mathbf{\Sigma} = \{\square, \blacksquare\}^N$. Therefore, if $x \in \mathbf{\Sigma}$ we write $x = (x_1, \ldots, x_i, \ldots, x_N)$ with $x_i \in \{\square, \blacksquare\}$. The agents are placed on a network $G = (\mathbf{N}, \mathbf{E})$ where $\mathbf{N}$ corresponds to the set of N agents and $\mathbf{E}$ is the set of agent connections (i, j). In the dynamic process implemented in the VM an agent i is chosen at random along with one of its neighboring agents j. If the states (x_i, x_j) are not equal already, agent i adopts the state of j (by setting $x_i = x_j$). The model has two absorbing states corresponding to the configurations with all agents holding the same opinion (consensus). The exit probabilities depend on the initial fraction of individuals in the respective state: for final consensus $x = (\blacksquare, \ldots, \blacksquare)$ and b_0 black agents at $t = 0$ we have $P_{\blacksquare}(b_0) = b_0$. Analytical results for the mean convergence times have been obtained for the complete graph [2, 6], for d–dimensional lattices [5, 11–13] as well as for networks with uncorrelated degree distributions [7, 8].

At the microscopic level of all possible configurations of agents the VM can be seen as an absorbing random walk on the N–dimensional hypercube with the two absorbing states $(\blacksquare, \ldots, \blacksquare)$ and $(\square, \ldots, \square)$. This is shown for the example of three agents in Fig. 1. Due to the sequential update process, from one interaction event to the other, only one agent changes or the configuration remains as it is (loops are not shown in Fig. 1). That is, non–zero transition probabilities exist only between configurations that are adjacent in the hypercube graph. Let us therefore call two agent configurations $y, x \in \mathbf{\Sigma}$ adjacent if they differ only with respect to a single agent i. We denote this by $x \overset{i}{\sim} y$.

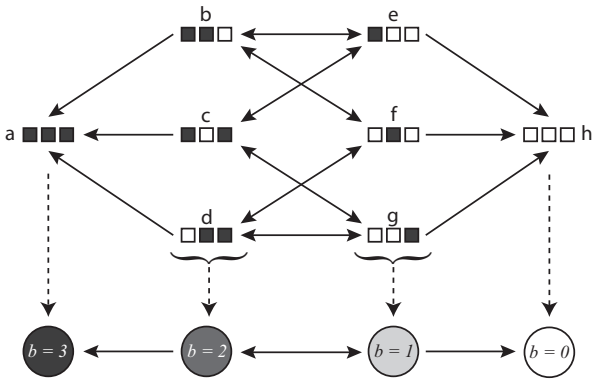

FIG. 1. The micro chain for the VM with 3 agents and its projection onto a birth–death random walk obtained by agglomeration of states with the same number of black agents b.

The transition probabilities associated with the arrows in Fig. 1 depend only on the probability with which an agent pair (i, j) is chosen such that their opinions (x_i, x_j) result in the respective transition. Different agent interaction networks have therefore a crucial effect on the transition probabilities of the micro chain. Let us denote by A the adjacency matrix of G such that $a_{ij} = 0$ states that there is no connection from agent i to j and $a_{ij} > 0$ that such a connection exists with a strength given by a_{ij}. We assume in the following the VM with link update dynamics such that an agent pair is selected for interaction by a random choice of a link (i, j) as is often convenient for heterogeneous graphs (see [10]: 601). The probability of finding an agent pair (i, j) is then $\frac{a_{ij}}{|\mathbf{E}|}$ where $|\mathbf{E}|$ is the sum over all edge weights. With these definitions the transition probability for adjacent

configurations with $x \overset{i}{\sim} y$ is given by

$$\hat{P}(x,y) = \frac{1}{|\mathbf{E}|} \sum_{j:(x_i \neq x_j)} a_{ij}. \quad (1)$$

This is the conjoint probability to find an agent pair (i,j) so that the ith agent has to change his opinion. Notice that for an undirected, unweighted graph the sum corresponds to the number of neighbors in conflict with i and $|\mathbf{E}|$ is the number of edges in G. See Ref. [2], Eq. (4), for similar a expression in a more general setting and Ref. [7], Eq. (9), for the VM with node update dynamics.

Notice that, at this level, the dynamics of the model is defined in the configuration space with 2^N states, which seeks to describe the dynamics of each agent in full detail. Following [2] we refer to this as *micro dynamics*. As the number of states increases as 2^N with the number of agents, it is not convenient to study the models dynamics at the micro level. It is often more convenient to reformulate the dynamics in terms of a macroscopic aggregate variable. In the VM, the number b of black agents i with $x_i = \blacksquare$ or the respective density are natural choices. Fig. 1 illustrates how several micro configurations with the same b are taken together to form a single macro state. Notice, that in the hypercube terminology b corresponds to the Hamming weight of a configuration.

It is well–known that the macro model obtained in terms of b fully describes the evolution of the micro model on the complete graph ([6]:3), but that "[n]onregular topologies have nontrivial effects on the ordering dynamics of the voter model" ([10]:601). The (probabilistic) reason for this is that the complete graph is the only topology for which the transition probabilities of the new macro process are uniquely defined. Only in that case configurations with the same Hamming weight b can be interchanged without changing the probability structure of the random walk on the hypercube (see [14]:24 or [15]:8). For instance, in the example shown in Fig.1, the probability (1) of a transition form configuration "e" to "h" is $\hat{P}(e,h) = (a_{12} + a_{13})/|\mathbf{E}|$, from "f" to "h" we have $\hat{P}(f,h) = (a_{21} + a_{23})/|\mathbf{E}|$ and for "g" to "h", $\hat{P}(g,h) = (a_{31} + a_{32})/|\mathbf{E}|$. While all these probabilities are equal for the complete graph (as $a_{ij} = 1 : \forall i,j$) they are not all equal if one or two connections are absent. For example, if $a_{23} = a_{32} = 0$ (see Fig. 2) we have $\hat{P}(e,h) \neq \hat{P}(f,h) = \hat{P}(g,h)$. In that case, the macro process shown in Fig. 1 is no longer a Markov chain and the formulation of the voter model in terms of b does not any longer completely describe the system.

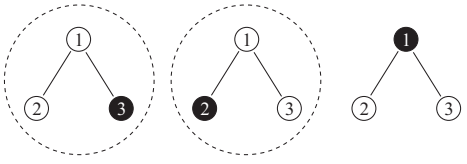

FIG. 2. The 3 different configurations "e", "f" and "g" of length 3 with one agent in ■ and two in □ ($b = 1$). The first two configurations are macroscopically equivalent.

Deriving such a complete description in form of a Markov chain requires a refinement of the aggregation procedure. We have to identify bundles of micro configurations that can be interchanged without changing the hypercubic micro chain. These bundles – henceforward called *partitions* – would then define a coarse–grained description of the micro process which still captures the dynamics in full detail.

For this refinement, we can make use of a theorem in Markov chain theory that gives necessary and sufficient conditions for partitioning the state space of Markov chains in such a way that the micro process projected onto that partition is still a Markov chain. This so–called lumpability theorem is presented in [16] Thm. 6.3.2. Let $\hat{p}_{xY}$ denote the conjoint probability for $x \in \mathbf{\Sigma}$ to go to elements $y \in Y$ where $Y \subseteq \mathbf{\Sigma}$ is a subset of the configuration space. The lumpability theorem states that a Markov chain $(\hat{P}, \mathbf{\Sigma})$ is *lumpable* with respect to a partition $\mathbf{X} = (X_1, \ldots, X_P)$ if for every two subsets X_k and X_l the sum $\hat{p}_{xX_l} = \sum_{y \in X_l} \hat{p}_{xy}$ is equal for all $x \in X_k$. Moreover, these common values form the transition probabilities $P(X_k \to X_l)$ for a new Markov chain $(P, \mathbf{X})$. For further convenience we define:

Definition I.1 *Two configurations x and x' are macroscopically equivalent if they belong to the same subset X_k.*

Because the transition probabilities (1) of the micro process depend on the adjacency relations in the agent graph G, the structure of G plays a decisive role in the construction of lumpable partitions of the configuration space $\mathbf{\Sigma}$. In fact, as will be shown in the sequel, the symmetries of G dictate an appropriate partitioning. Let us consider again the example of three agents now connected as shown in Fig. 2 ($a_{23} = a_{32} = 0$). This graph has a reflection symmetry such that the nodes 2 and 3 can be permuted without affecting the connectivity structure. This symmetry imposes a symmetry in the hypecubic graph associated to the micro process such that the configurations "f" and "g" with $b = 1$ and respectively "b" and "c" with $b = 2$ can be permuted without affecting the transition probabilities. This motivates the notions of *macroscopic equivalence* because the macro evolution of the model $(P, \mathbf{X})$ is the same for these configurations and we can lump them into a single macro state.

More formally: the symmetries in G are captured by its automorphism group, denoted here as $Aut(G)$ or simply $\mathcal{G}$ (see e.g., [17, 18]). The group $\mathcal{G}$ is defined as the set of all isomorph mappings of a graph onto itself. In other words, $\mathcal{G}$ is the set of all permutations σ on the set of nodes (here the set of agents $\mathbf{N}$) such that every two nodes/agents linked by a_{ij} in G are linked by the same value in the image σG of G, that is: $a_{ij} = a_{\sigma i \sigma j} : \forall (i,j)$. It is well–known (cf. [17]:38) that the automorphism group $\mathcal{G}$ acting on the set of agents $\mathbf{N} = \{1, 2, \ldots, N\}$ induces an equivalence relation on $\mathbf{N}$ such that two labeled graphs G_1, G_2 are called $\mathcal{G}$*–equivalent* if and only if there is a $\sigma \in \mathcal{G}$ such that $G_1 = \sigma G_2$. The equivalence classes with respect to $\mathcal{G}$ are usually called $\mathcal{G}$*–orbits* and the equivalence class that contains the labeled graph G_1 is defined by $\mathcal{G} \circ G_1 := \{\sigma G_1 : \sigma \in \mathcal{G}\}$.

A configuration x of the VM can be seen as a graph of

size N labeled using a set of two labels $\mathbf{S} = \{\square, \blacksquare\}$. Let us define permutations on $x = (x_1, \ldots, x_i, \ldots, x_N) \in \mathbf{\Sigma}$ by $\sigma(x) = (x_{\sigma 1}, \ldots, x_{\sigma i}, \ldots, x_{\sigma N})$. As before the automorphism group of the agent network $\mathcal{G}$ defines equivalence classes of configurations such that $\mathcal{G} \circ x := \{\sigma(x) : \sigma \in \mathcal{G}\}$ is the set of all configurations that are $\mathcal{G}$–equivalent to x. The following Proposition states that these configuration then are macroscopically equivalent:

Proposition I.1 *Let $\mathcal{G}$ be the automorphism group of the agent network G. Two configurations x and x' are macroscopically equivalent if there exists (at least) one permutation $\sigma \in \mathcal{G}$ such that $x = \sigma(x')$. The respective partition $\mathbf{X}$ is defined by the orbits that $\mathcal{G}$ induces on $\mathbf{\Sigma}$.*

Proof. Consider a graph G and denote the permutations in the automorphism group by $\sigma_i \in \mathcal{G}, i = 0, \ldots, |\mathcal{G}|$. Suppose we know (at least) one configuration (the generator) $x^j \in \mathbf{\Sigma}$ for each $X_j \subset \mathbf{\Sigma}$ and construct the partition $\mathbf{X} = (X_1, \ldots, X_j, \ldots)$ by

$$X_j = \mathcal{G} \circ x^j = \bigcup_{i=0}^{|\mathcal{G}|} \sigma_i(x^j). \tag{2}$$

A necessary and sufficient condition for lumpability is that the $\hat{p}_{xY}$ be the same for all $x \in X$ and any pair of subsets X and Y (Thm. 6.3.2 in [16]). Due to the micro structure of the VM, this is satisfied whenever $\hat{p}_{xy} = \hat{p}_{\sigma(x)\sigma(y)}$ for any $\sigma \in \mathcal{G}$ (cf. Prop. 3.1 in [2]). For the case that $x \overset{i}{\sim} y$ we know that $x_j = y_j$ for all j except i and that the transition requires the choice of an edge $(i, .)$. Denoting $x_i = s$ and $y_i = \bar{s}$ we rewrite Eq. (1) as

$$\hat{p}_{xy} = \frac{1}{|\mathbf{E}|} \sum_{j:(x_j = \bar{s})} a_{ij}. \tag{3}$$

If $x \overset{i}{\sim} y$ it is easy to show that $\sigma(x) \overset{\sigma i}{\sim} \sigma(y)$ and we know that $s = \sigma(x_{\sigma i}) \neq \sigma(y_{\sigma i}) = \bar{s}$. The transition therefore requires the choice of an edge $(\sigma i, .)$. We obtain

$$\hat{p}_{\sigma(x)\sigma(y)} = \frac{1}{|\mathbf{E}|} \sum_{k:(\sigma(x_k) = \bar{s})} a_{\sigma i k}. \tag{4}$$

Given an arbitrary configuration x, for any j with $x_j = \bar{s}$ we have one $k = \sigma j$ with $\sigma(x_k) = \bar{s}$ because $x_j = \bar{s} \Leftrightarrow \sigma(x_{\sigma j}) = \bar{s}$. That is, the summations in Eq.(3) and (4) are equal for any σ for which $a_{ij} = a_{\sigma i \sigma j}$. This is true by definition for all permutations in the automorphism group $\sigma \in \mathcal{G}$. ■

Consider as a first example the complete graph (K_N) which has been discussed previously by several authors, including [2, 6]. The automorphism group of K_N is the symmetric group $(Aut(K_N) = \mathcal{S}_N)$, that is, the group of all permutations of N agents. Therefore, for any two configurations x, x' with equal b there is a $\sigma \in \mathcal{S}_N$ such that $x = \sigma(x')$. Hence, an equivalent aggregate value b implies macroscopic equivalence. The fact that this is only true for complete graph, or respectively homogeneous mixing, underlines the theoretical importance of this topology.

The associated macro process on the partition $\mathbf{X} = (X_0, \ldots, X_b, \ldots X_N)$ is a simple birth–death random walk with $P(X_b \to X_{b\pm 1}) = \frac{b(N-b)}{N^2}$, $P(X_b \to X_b) = \frac{b^2 + (N-b)^2}{N^2}$. In [2] we have derived a closed–form expression for the fundamental matrix of that Markov chain for arbitrary N. Encoding the recurrence and hitting times of the system, this provides all the information to characterize the distribution of convergence times. For instance, for the VM starting in a state with b black agents the mean time to reach consensus is given by

$$\tau_b = N \left[\sum_{j=1}^{b-1} \frac{(N-b)}{(N-j)} + 1 + \sum_{j=b+1}^{N-1} \frac{b}{j} \right]. \tag{5}$$

We also considered in [2] the VM with more opinions and confirm by lumpability arguments that the convergence times are as in the binary case, which was shown previously in Ref. [6].

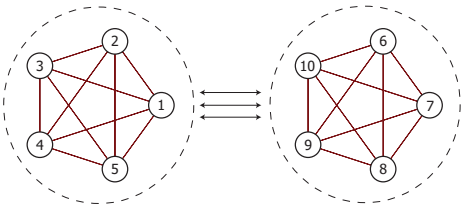

FIG. 3. A two–component graph with two homogeneous sub–populations.

The method also yields a very intuitive aggregation for structured populations as shown in Fig. 3. Consider a graph with two completely connected subpopulations of size L and M $(L + M = N)$ and homogeneous influences across the two clusters. In population genetics this is called the island model [19]. The automorphism group of such a structure is $\mathcal{G} = (1 \ldots L)(L+1 \ldots N)$. There are exactly $L!M!$ automorphisms because $\mathcal{G}$ is composed of the symmetric group $\mathcal{S}_L$ and $\mathcal{S}_M$ acting on the homogeneous subgraphs. Therefore, the state of the system is completely characterized by the numbers b_L and b_M of black agents in either subgraph because all configurations x and y with $b_L(x) = b_L(y) \cap b_M(x) = b_M(y)$ are within the same equivalence class. As $0 \leq b_L \leq L$ and $0 \leq b_M \leq M$ the aggregation defines a Markov chain with $(L+1)(M+1)$ states which is still very small compared to the number of $2^{(L+M)}$ micro configurations. Notice that this generalizes naturally to a larger number of subgraphs. Notice also that the multipartite graphs studied in [7] fall into this category and that the authors used the respective sub–densities in their mean–field description.

The cases of homogeneous graphs or graphs composed of homogeneous sub–graphs are particularly simple because it is intuitively clear which configurations can be aggregated. However, Prop. I.1 generalizes to networks with arbitrary automorphisms which we illustrate at the example of the ring graph. The ring with nearest neighbor interactions is defined by $a_{ij} = 1$ whenever $|i-j| = 1$ and $a_{N1} = a_{1N} = 1$. It possesses an invariance with respect to translations such that the automorphism group consists of all cyclic shifts of agents $\mathcal{G} = \mathcal{C}_N$. Let us define

the simple shift as $\sigma_1(1,2,\ldots,N) = (N,1,2,\ldots,N-1)$. The automorphism group is the set of repeated applications of σ_1: $\mathcal{G} = \{\sigma_k = \sigma_1^k : k = 1,\ldots,N\}$ where $\sigma_1^N = \sigma_0$ is the identity permutation. Notice that translational symmetries of this kind also play an important role in the determination of the relevant dimensions of spin rings [20] and that there are interesting parallels in between the two problems.

Consider a ring of five agents ($N = 5$) with $2^5 = 32$ micro states. For $x = (\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare)$ it is clear that $\sigma_k(x) = x$ for all k. That is, $x = (\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare)$ with $b = 5$ constitutes a class of its own. For $b = 4$, we may use $x^1 = (\square\blacksquare\blacksquare\blacksquare\blacksquare)$ as a generator in Eq. (2). As all 5 configurations with $b = 4$ can be obtained by a shift of x^1, all of them are in the same equivalence class (e.g., $\sigma_2(x^1) = (\blacksquare\blacksquare\square\blacksquare\blacksquare)$). The 10 configurations with $b = 3$ cannot be lumped into the same macro state. There are two classes differentiated by the distance of zero or one in between the two black agents (see Fig. 4). Using the two configurations shown in Fig. 4 as generators, $\mathcal{G} \circ x^j$ yields two equivalence classes each containing five micro states. The cases $b = 2, 1, 0$ follow by symmetry so that all in all the dimension $\mathbf{X}$ of macro chain is reduced to 8.

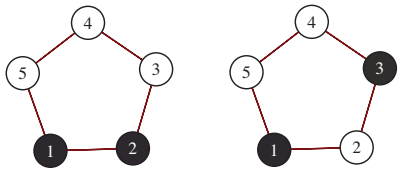

FIG. 4. Two configurations with equal b which are not macroscopically equivalent for the ring with $N = 5$.

In the general case of N agents we can in principle proceed in the same way. However, the number of macro states will increase considerably with the system size. We finish this Letter with a quantification of this number for the ring for which we can use a well–known enumeration theorem due to Pólya (see [17]:35-45, Eqs.(2.2.10) and (2.4.15) in particular). According to this, the number of macro states is

$$|\mathbf{X}| = \frac{1}{N}\sum_{k|N} \varphi(k) 2^{\frac{N}{k}} \tag{6}$$

where $\varphi(k)$ is the Euler φ–function and the sum is over the divisors $k|N$ of N. As an approximation we have $|\mathbf{X}| \approx 2^N/N$. Hence, an explicit solution of the macro chain will be possible only for very small systems.

In conclusion, this Letter describes a way to reduce the state space of the VM by exploiting all the dynamical redundancies that have its source in the agent network. In the aggregation of micro states into macro atoms following this proposal no information about the details of the dynamical process is omitted. However, it is clear that in heterogeneous interaction substrates with a small number of automorphisms the coarse–graining is limited because only a few micro states are macroscopically equivalent and can be lumped. On the other hand, the method informs us in this way about the complexity of a system introduced by non–trivial interaction relations. Even in a model as simple as the VM, the behavior of whole system is not completely described by summation over its elements (aggregation in terms of b), because non–trivial dynamical effects may emerge at the macro level. In fact, Markov projections as discussed here in the context of the VM may provide explicit knowledge about the (in)compressibility of computational models which is one of the key concepts in the area of computational emergence ([21, 22]).

The probabilistic setting we adopt allows to relate microscopic agent dynamics to the macro evolution of aggregate observable variables and shows how network heterogeneities translate into heterogeneities in the dynamical structure of the model. A characterization of the dynamical effects that may emerge at the macro level will be addressed by future work.

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